Definitions: Tautological entailment and FDE

Philosophical Logic 2025/2026

1 Tautological entailment

Definition 1.1 (Literals and primitive formulas). *Let* Var *be a set of propositional variables*

1. *A* literal *is either a propositional variable* $p \in \mathsf{Var}$ *or its negation* $\neg p$
2. *A* primitive conjunction *is a finite non-empty conjunction of literals, i.e. any formula of the form*
$$\ell_1 \wedge \cdots \wedge \ell_n$$
with $n \geq 1$ *and each* ℓ_i *a literal*
3. *A* primitive disjunction *is a finite non-empty disjunction of literals, i.e. any formula of the form*
$$\ell_1 \vee \cdots \vee \ell_n$$
with $n \geq 1$ *and each* ℓ_i *a literal*

Definition 1.2 (Elementary tautological entailment). *Let* ϕ *and* ψ *be formulas. We say that* ϕ elementarily tautologically entails ψ *and write* $\phi \vDash_{ET} \psi$ *if the following two conditions hold*

1. ϕ *is a primitive conjunction and* ψ *is a primitive disjunction*
2. *some literal occurs both in* ϕ *and in* ψ

Definition 1.3 (Tautological entailment). *Let* ϕ *and* ψ *be arbitrary formulas. We say that* ϕ tautologically entails ψ *and write* $\phi \vDash_T \psi$ *if the following holds*

1. ϕ *can be rewritten into a logically equivalent formula in disjunctive normal form*
$$\phi_1 \vee \cdots \vee \phi_n$$
where each ϕ_i *is a primitive conjunction*
2. ψ *can be rewritten into a logically equivalent formula in conjunctive normal form*
$$\psi_1 \wedge \cdots \wedge \psi_m$$
where each ψ_j *is a primitive disjunction*

3. *for every $i \in \{1, \dots, n\}$ and every $j \in \{1, \dots, m\}$ we have $\phi_i \vDash_{ET} \psi_j$*

Definition 1.4 (Normal Form Conversion). *Let ϕ, ψ, χ be arbitrary formulas. In order to convert formulas into disjunctive normal form and conjunctive normal form for step 1 and step 2 of Definition 1.3, one can use the following standard equivalence schemata.*

Commutation

$$\phi \wedge \psi \iff \psi \wedge \phi$$
$$\phi \vee \psi \iff \psi \vee \phi$$

Association

$$(\phi \wedge \psi) \wedge \chi \iff \phi \wedge (\psi \wedge \chi)$$
$$(\phi \vee \psi) \vee \chi \iff \phi \vee (\psi \vee \chi)$$

Distribution

$$\phi \wedge (\psi \vee \chi) \iff (\phi \wedge \psi) \vee (\phi \wedge \chi)$$
$$\phi \vee (\psi \wedge \chi) \iff (\phi \vee \psi) \wedge (\phi \vee \chi)$$

Double negation

$$\neg\neg\phi \iff \phi$$

De Morgan's laws

$$\neg(\phi \wedge \psi) \iff \neg\phi \vee \neg\psi$$
$$\neg(\phi \vee \psi) \iff \neg\phi \wedge \neg\psi$$

2 Axiomatic First Degree Entailment

Definition 2.1 (Axiomatic system for $\vdash_T$). *The relation $\vdash_T$ of derivability in the logic of tautological entailment is the smallest relation on formulas that contains the following axioms and is closed under the following rules*

Axioms

Conjunction

$$\phi \wedge \psi \vdash_T \phi$$
$$\phi \wedge \psi \vdash_T \psi$$

Disjunction

$$\phi \vdash_T \phi \vee \psi$$
$$\psi \vdash_T \phi \vee \psi$$

Distribution

$$\phi \wedge (\psi \vee \chi) \vdash_T (\phi \wedge \psi) \vee \chi$$

Negation (double negation)

$$\phi \vdash_T \neg\neg\phi$$

$$\neg\neg\phi \vdash_T \phi$$

Rules of inference

Transitivity *From* $\phi \vdash_T \psi$ *and* $\psi \vdash_T \chi$ *infer* $\phi \vdash_T \chi$

Conjunction *From* $\phi \vdash_T \psi$ *and* $\phi \vdash_T \chi$ *infer* $\phi \vdash_T \psi \wedge \chi$

Disjunction *From* $\phi \vdash_T \chi$ *and* $\psi \vdash_T \chi$ *infer* $\phi \vee \psi \vdash_T \chi$

Negation (contraposition) *From* $\phi \vdash_T \psi$ *infer* $\neg\psi \vdash_T \neg\phi$

3 Four-valued semantics (FDE)

Definition 3.1 (Truth values)**.** *The four truth values of FDE are the four subsets of* $\{1, 0\}$

$$\{\{1\}, \{0\}, \emptyset, \{1, 0\}\}$$

$\{1\}$ *represents being true only,* $\{0\}$ *being false only,* $\emptyset$ *being neither true nor false, and* $\{1, 0\}$ *being both true and false*

Definition 3.2 (FDE-valuations)**.** *An* FDE-valuation *is a function* v *that assigns to each propositional variable a value in* $\{\{1\}, \{0\}, \emptyset, \{1, 0\}\}$ *and is extended inductively to all formulas as follows*

$$\begin{aligned}
1 \in v(\neg\phi) &\text{ iff } 0 \in v(\phi)\\
0 \in v(\neg\phi) &\text{ iff } 1 \in v(\phi)\\
1 \in v(\phi \wedge \psi) &\text{ iff } 1 \in v(\phi) \text{ and } 1 \in v(\psi)\\
0 \in v(\phi \wedge \psi) &\text{ iff } 0 \in v(\phi) \text{ or } 0 \in v(\psi)\\
1 \in v(\phi \vee \psi) &\text{ iff } 1 \in v(\phi) \text{ or } 1 \in v(\psi)\\
0 \in v(\phi \vee \psi) &\text{ iff } 0 \in v(\phi) \text{ and } 0 \in v(\psi)
\end{aligned}$$

Definition 3.3 (Four-valued semantic consequence)**.** *Let* ϕ *and* ψ *be formulas. We say that* ϕ FDE-semantically entails ψ *and write* $\phi \vDash_{FDE} \psi$ *if for every FDE-valuation* v *the following two conditions hold*

1. *if* $1 \in v(\phi)$*, then* $1 \in v(\psi)$
2. *if* $0 \in v(\psi)$*, then* $0 \in v(\phi)$